MOTION OF A CIRCULAR CYLINDER IN A VIBRATING LIQUID

V. L. Sennitskii

UDC 532.582

Qualitative results of experiments involving vibrating liquids and solids were presented in [1]. In particular, it was shown that a body situated in a cylindrical vessel with a liquid having a density greater than that of the solid may sink if the vessel performs oscillations along its axis. In connection with this, the present study will investigate the planar problem of motion in a gravitational force field of a circular cylinder situated in an ideal incompressible liquid, limited externally by the planar surface of the oscillating wall (see Fig. 1). The liquid and cylinder are initially at rest. At subsequent times the liquid flow is potential and symmetric about the x axis, and the cylinder moves in translation. Conditions will be found under which cylinders with densities less than the density of the surrounding liquid sink rather than float upward.

1. Let x, y be an inertial rectangular coordinate system in the plane of the flow; **i** and **j** are unit vectors directed along the x and y axes; t is time; a is the cylinder radius; $O(L, O)$ is the point of interaction of the flow plane with the axis of the cylinder; h is the distance from the point O to the line of intersection between the flow plane and the wall surface ($h > a$); h_0 is the value of h at t = 0; H = L − h; $\hat{x} = x - H$; $r = \sqrt{(\hat{x} - h)^2 + y^2}$; θ is the angle at the point O between the vectors **i** and $(\hat{x} - h)\mathbf{i} + y\mathbf{j}$; ρ_c is the density of the cylinder; ρ_l is the density of the liquid; f is an arbitrary function of t; $\mathbf{g} = -g\mathbf{i}$ is the acceleration of gravity.

We will consider liquid flow and cylinder motion relative to a coordinate system $\hat{x}$, y, fixed to the wall. The potential Φ of the flow velocity, the pressure p, and distance h satisfy the following equations and conditions:

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left(\frac{\partial\Phi}{\partial\hat{x}}\right)^2 + \frac{1}{2}\left(\frac{\partial\Phi}{\partial y}\right)^2 + \frac{p}{\rho_{\text{ж}}} + \left(g + \frac{d^2H}{dt^2}\right)\hat{x} = f; \qquad (1.1)$$

$$\partial^2\Phi/\partial\hat{x}^2 + \partial^2\Phi/\partial y^2 = 0; \qquad (1.2)$$

$$\partial\Phi/\partial\hat{x} = 0 \text{ for } \hat{x} = 0; \qquad (1.3)$$

$$\partial\Phi/\partial y = 0 \text{ for } y = 0,\ 0 \leqslant \hat{x} \leqslant h - a \text{ and } y = 0,\ \hat{x} \geqslant h + a; \qquad (1.4)$$

$$\frac{\partial\Phi}{\partial\hat{x}}\cos\theta + \frac{\partial\Phi}{\partial y}\sin\theta = \frac{dh}{dt}\cos\theta \quad \text{for} \quad r = a; \qquad (1.5)$$

$$\partial\Phi/\partial\hat{x} \to 0,\ \partial\Phi/\partial y \to 0 \text{ at } \hat{x}^2 + y^2 \to \infty; \qquad (1.6)$$

$$d^2h/dt^2 = F/(\pi a^2\rho_c) - g - d^2H/dt^2; \qquad (1.7)$$

$$h = h_0,\ dh/dt = 0 \text{ at } t = 0, \qquad (1.8)$$

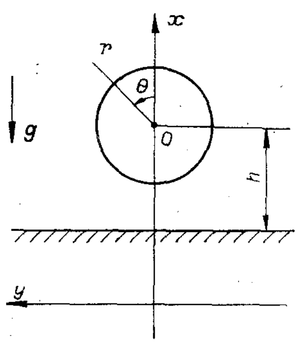

Fig. 1

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 19-23, September-October, 1985. Original article submitted April 10, 1984.

 0021-8944/85/2605-0620$09.50

where

$$F = -a \int_{-\pi}^{\pi} \dot{p}|_{r=a} \cos\theta\, d\theta. \tag{1.9}$$

2. We will consider the problem of Eqs. (1.2)-(1.6). Transforming in Eqs. (1.2)-(1.6) to bipolar coordinates η, ξ, related to $\hat{x}$, y by the expressions

$$\hat{x} = \sqrt{h^2 - a^2} \frac{\operatorname{sh} \eta}{\operatorname{ch} \eta - \cos \xi}, \quad y = \sqrt{h^2 - a^2} \frac{\sin \xi}{\operatorname{ch} \eta - \cos \xi}, \tag{2.1}$$

we obtain

$$\partial^2\Phi/\partial\eta^2 + \partial^2\Phi/\partial\xi^2 = 0; \tag{2.2}$$

$$\partial\Phi/\partial\eta = 0 \text{ at } \eta = 0,\ \xi \neq 0; \tag{2.3}$$

$$\partial\Phi/\partial\xi = 0 \text{ at } \xi = \pm\pi \text{ and } \xi = 0,\ \eta \neq 0; \tag{2.4}$$

$$\frac{\partial\Phi}{\partial\eta} = a\sqrt{h^2 - a^2} \frac{a - h\cos\xi}{(h - a\cos\xi)^2} \frac{dh}{dt} \text{ at } \eta = \eta_0; \tag{2.5}$$

$$\begin{gathered} (1 - \operatorname{ch}\eta\cos\xi)\frac{\partial\Phi}{\partial\eta} - \operatorname{sh}\eta\sin\xi\frac{\partial\Phi}{\partial\xi} \to 0, \\ -\operatorname{sh}\eta\sin\xi\frac{\partial\Phi}{\partial\eta} + (\operatorname{ch}\eta\cos\xi - 1)\frac{\partial\Phi}{\partial\xi} \to 0 \text{ for } \eta^2 + \xi^2 \to 0, \end{gathered} \tag{2.6}$$

where

$$\eta_0 = \ln \frac{h + \sqrt{h^2 - a^2}}{a}. \tag{2.7}$$

The functions $\eta = \eta(\hat{x}, y)$, $\xi = \xi(\hat{x}, y)$ map the flow region cut off by the section $0 \leqslant \hat{x} \leqslant h - a$, $y = 0$ into the triangle $0 \leqslant \eta \leqslant \eta_0$, $-\pi \leqslant \xi \leqslant \pi$. Separating the variables and making use of the equality

$$\int_{-\pi}^{\pi} \frac{h\cos\xi - a}{(h - a\cos\xi)^2} \cos m\xi\, d\xi = \frac{2m\pi}{a} e^{-m\eta_0} \text{ for } m = 0, 1, \ldots, \tag{2.8}$$

we find the following solution of the problem of Eqs. (2.2)-(2.6):

$$\Phi = -2\sqrt{h^2 - a^2} \frac{dh}{dt} \sum_{m=1}^{\infty} \frac{\operatorname{ch} m\eta \cos m\xi}{e^{m\eta_0} \operatorname{sh} m\eta_0} + \varphi, \tag{2.9}$$

where φ is an arbitrary function of t. Equations (2.1), (2.7), (2.9) define the solution of Eqs. (1.2)-(1.6).

We will substitute Eq. (2.9) into Eq. (1.1). Using the relationship obtained and Eqs. (1.9), (2.1), (2.5), (2.8), we find

$$F = \pi a^2 \rho_{ж} \left[g + \frac{d^2H}{dt^2} + f_1 \frac{d^2h}{dt^2} + f_2 a^{-1} \left(\frac{dh}{dt} \right)^2 \right], \tag{2.10}$$

where

$$f_1 = -4\operatorname{sh}^2\eta_0 \sum_{m=1}^{\infty} a_m, \quad a_m = m e^{-2m\eta_0} \operatorname{cth} m\eta_0; \tag{2.11}$$

$$\begin{gathered} f_2 = 2\operatorname{sh}\eta_0 \sum_{m=1}^{\infty} b_m + 4\operatorname{sh}\eta_0(\operatorname{ch}^2\eta_0 + 1)\sum_{m=1}^{\infty} c_m - \frac{\operatorname{ch}\eta_0}{\operatorname{sh}^2\eta_0}, \\ b_m = m e^{-2m\eta_0} \operatorname{cth} m\eta_0 [m \operatorname{cth} m\eta_0 + (m+1) e^{-\eta_0} \operatorname{ch}\eta_0 \operatorname{cth}(m+1)\eta_0], \\ c_m = m e^{-2m\eta_0}(m - \operatorname{cth}\eta_0)\operatorname{cth} m\eta_0. \end{gathered} \tag{2.12}$$

The series in Eqs. (2.11), (2.12) and their residues satisfy the following inequalities:

$$\sum_{m=N}^{\infty} a_m < \left(N + \frac{1}{2}\right) \operatorname{cth}^3 \frac{\varepsilon}{2} e^{-2N\eta_0},$$

$$\sum_{m=N}^{\infty} b_m < 2(N+2)^2 \operatorname{cth}^5 \frac{\varepsilon}{2} e^{-2N\eta_0},$$

$$\left|\sum_{m=N}^{\infty} c_m\right| < (N+2)^2 \operatorname{cth}^4 \frac{\varepsilon}{2} e^{-2N\eta_0} \quad \text{при} \quad \eta_0 > \varepsilon, \tag{2.13}$$

where N = 1, 2,...; ε is any positive number.

We note that the actions performed on the series to obtain Eqs. (2.9), (2.10) are admissible since the corresponding sufficient conditions are satisfied [2].

3. Let T be the period of the wall oscillations;

$$H = A_0 + \sum_{m=1}^{\infty} \left(A_m \cos 2m\pi \frac{t}{T} + B_m \sin 2m\pi \frac{t}{T}\right), \tag{3.1}$$

A_0, A_m, B_m are constants;

$$A_0 + \sum_{m=1}^{\infty} A_m = 0, \quad \sum_{m=1}^{\infty} mB_m = 0; \tag{3.2}$$

A is the largest value of |H|. According to Eqs. (2.7), (2.11)-(2.13) we have

$$f_1 = -\left[1 + \frac{a^2}{2h^2} + O\left(\frac{a^4}{h^4}\right)\right], \quad f_2 = \frac{a^3}{2h^3}\left[1 + O\left(\frac{a^2}{h^2}\right)\right] \text{ for } \frac{a}{h} \to 0. \tag{3.3}$$

Using Eqs. (2.10), (3.1), (3.3), we find that for L independent of time, $A \gtrless a$ and $A/L \to 0$, the equation

$$F = \pi a^2 \rho_l g \left[\bar{f} + \tilde{f} + O\left(\frac{a^2 A^3}{L^4 g T^2}\right)\right],$$

is satisfied, where $\bar{f} = 1 - \frac{\pi^2 k A^2 a^2}{L^3 g T^2}$, $k = A^{-2} \sum_{m=1}^{\infty} m^2 (A_m^2 + B_m^2)$; $\tilde{f} = \sum_{m=1}^{\infty} \left(\tilde{A}_m \cos 2m\pi \frac{t}{T} + \tilde{B}_m \sin 2m\pi \frac{t}{T}\right)$,

$\tilde{A}_m$, $\tilde{B}_m$ are constants. For $\rho_c < \rho_l$, the sum of the time-independent force $\pi a^2 \rho_l g \bar{f}$ and the gravitational force $-\pi a^2 \rho_c g$ acting on a unit length of the cylinder is negative if

$$\pi^2 k A^2 a^2 \rho_l / [L^3(\rho_l - \rho_c) g T^2] > 1.$$

This result indicates that with sufficiently low values of T, a circular cylinder with density less than that of the density of the liquid in which it is located may sink rather than float.

4. We will consider the problem of Eqs. (1.7), (1.8), (2.7), (2.10)-(2.12). At $\rho_c = \rho_l$ it has the solution $h = h_0$. Let

$$\rho_c \neq \rho_l, \ a/h_0 \to 0, \ A/h_0 \to 0, \ \mu \to 0,$$
$$aA/(\mu h_0^2) = \alpha, \ A/a > \beta, \ h/h_0 > \gamma, \tag{4.1}$$

where $\mu = h_0^{-1/2} g^{1/2} T$; α, β, γ are positive numbers ($\gamma < 1$). Using Eqs. (1.7), (2.10), (3.3) we obtain the following approximate equation:

$$\frac{d^2h}{dt^2} = -\varkappa\left(g + \frac{d^2H}{dt^2}\right)\left(1 - \lambda\frac{a^2}{h^2}\right) + \lambda\frac{a^2}{h^3}\left(\frac{dh}{dt}\right)^2, \tag{4.2}$$

where

$$\varkappa = (\rho_c - \rho_l)/(\rho_c + \rho_l); \ \lambda = \rho_l / [2(\rho_c + \rho_l)].$$

We will solve the problem of Eqs. (1.8), (4.2) by the averaging method of [3, 4]. Using the substitution

$$h = h_0 z - \varkappa H + \varkappa\lambda a^2 H/(h_0 z)^2, \ t = T\tau, \tag{4.3}$$

we reduce Eq. (4.2) to a system of equations in standard form:

$$dz/d\tau = \mu Z,$$
$$dZ/d\tau = \mu\varkappa\{\alpha^2\varkappa\lambda A^{-2}[2H d^2H/d\tau^2 + (dH/d\tau)^2] z^{-3} - 1\}. \tag{4.4}$$

Using Eqs. (4.1), (4.3) and the equality obtained below, defining the dependence of z on τ, it can be verified that the terms dropped in transition from Eq. (4.2) to Eq. (4.4) are small in comparison to the terms maintained in Eq. (4.4). In the first approximation of the averaging method

$$z = \bar{z},\ Z = \bar{Z}, \tag{4.5}$$

where $\bar{z}$, $\bar{Z}$ is a solution of the system

$$d\bar{z}/d\tau = \mu\bar{Z},\ d\bar{Z}/d\tau = -\mu\varkappa(2\pi^2\alpha^2\varkappa\lambda k\bar{z}^{-3} + 1), \tag{4.6}$$

obtainable from Eq. (4.4) by averaging the latter over the dimensionless time τ (see [3, 4]). Limiting ourselves to this approximation and using Eqs. (4.5), (4.6), we obtain

$$d^2z/d\tau^2 = \mu^2\varkappa(\sigma z^{-3} - 1), \tag{4.7}$$

where

$$\sigma = -2\pi^2\alpha^2\varkappa\lambda k.$$

According to Eqs. (1.8), (3.1), (3.2), (4.3), we have

$$z = 1,\ dz/d\tau = 0 \text{ for } \tau = 0. \tag{4.8}$$

Integrating Eq. (4.7) and using Eq. (4.8) we find

$$\frac{1}{\sqrt{2}}\int_1^z \frac{s\,ds}{\sqrt{\varkappa(1-s)\left(s^2 - \frac{\sigma}{2}s - \frac{\sigma}{2}\right)}} = \begin{cases} \mu\tau & \text{for } 0 < \sigma < 1, \\ -\mu\tau & \text{for } \sigma < 0 \text{ and } \sigma > 1; \end{cases}$$
$$z = 1 \text{ for } \sigma = 1. \tag{4.9}$$

We note that

$$\text{for } \sigma > 0\ \rho_c < \rho_l, \qquad \text{for } \sigma < 0\ \rho_c > \rho_l. \tag{4.10}$$

It follows from Eq. (4.9) that at $0 < \sigma < 1$ z increases monotonically, while at $\sigma < 0$ and $\sigma > 1$ z decreases monotonically with increase in τ. Thus, if a/h_0, A/h_0, μ are small, A/a is not small in comparison to unity, and $\sigma > 1$, then according to Eqs. (4.3), (4.9), (4.10) a circular cylinder of density less than the density of the liquid in which it is located sinks.

LITERATURE CITED

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